

# On the branching of the quasinormal resonances of near-extremal Kerr black holes

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It has recently been shown by Yang et. al. [Phys. Rev. D **87**, 041502(R) (2013)] that rotating Kerr black holes are characterized by *two* distinct sets of quasinormal resonances. These two families of quasinormal resonances display qualitatively different asymptotic behaviors in the extremal ( $a/M \rightarrow 1$ ) black-hole limit: The zero-damping modes (ZDMs) are characterized by relaxation times which tend to infinity in the extremal black-hole limit ( $\Im\omega \rightarrow 0$  as  $a/M \rightarrow 1$ ), whereas the damped modes (DMs) are characterized by non-zero damping rates ( $\Im\omega \rightarrow \text{finite-values}$  as  $a/M \rightarrow 1$ ). In this paper we refute the claim made by Yang et. al. that co-rotating DMs of near-extremal black holes are restricted to the limited range  $0 \leq \mu \lesssim \mu_c \approx 0.74$ , where  $\mu \equiv m/l$  is the dimensionless ratio between the azimuthal harmonic index  $m$  and the spheroidal harmonic index  $l$  of the perturbation mode. In particular, we use an analytical formula originally derived by Detweiler in order to prove the existence of DMs (damped quasinormal resonances which are characterized by *finite*  $\Im\omega$  values in the  $a/M \rightarrow 1$  limit) of near-extremal black holes in the  $\mu > \mu_c$  regime, the regime which was claimed by Yang et. al. not to contain damped modes. We show that these co-rotating DMs (in the regime  $\mu > \mu_c$ ) are expected to characterize the resonance spectra of rapidly-rotating (near-extremal) black holes with  $a/M \gtrsim 1 - 10^{-9}$ .

## I. INTRODUCTION

Perturbed black holes display a unique pattern of damped oscillations, known as quasinormal resonances, which characterize the relaxation phase of the black-hole spacetime. The spectrum of quasinormal resonances reflects the physical parameters (such as mass, charge, and angular momentum) of the black-hole spacetime.

The complex quasinormal resonances correspond to perturbation fields which propagate in the black-hole spacetime with the physically motivated boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the black-hole horizon [1]. These boundary conditions single out a discrete set,  $\{\omega^{\text{QNM}}(n; m, l)\}_{n=0}^{\infty}$ , of complex black-hole resonances for each perturbation mode (here  $m$  and  $l$  are the azimuthal harmonic index and the spheroidal harmonic index of the wave field, respectively).

In a very interesting paper, Yang et. al. [2] have recently studied numerically the quasinormal spectrum of near-extremal (rapidly-rotating) Kerr black holes. The authors of [2] have reached the remarkable conclusion that these rapidly-rotating black holes are characterized by *two* qualitatively distinct sets of quasinormal resonances:

- Zero-damping modes (ZDMs), which are characterized by the asymptotic property [3]

$$\Im\omega^{\text{ZDM}}(\tau \rightarrow 0) \rightarrow 0, \quad (1)$$

and

- Damped modes (DMs), which are characterized by the asymptotic property

$$\Im\omega^{\text{DM}}(\tau \rightarrow 0) \rightarrow \text{finite values}. \quad (2)$$

Here

$$\tau \equiv \frac{r_+ - r_-}{r_+} \quad (3)$$

is the dimensionless Bekenstein-Hawking temperature of the black hole [4], where  $r_{\pm} \equiv M \pm (M^2 - a^2)^{1/2}$  are the black-hole (event and inner) horizons. This dimensionless temperature approaches zero in the extremal  $a \rightarrow M$  ( $r_- \rightarrow r_+$ ) limit of rapidly-rotating black holes.

## II. THE ERRONEOUS CLAIM MADE IN [2] AND DETWEILER'S DAMPED RESONANCES

It has been asserted in Ref. [2] that the ZDMs (1) exist for all co-rotating modes ( $m \geq 0$ ) [3], whereas the DMs (2) exist for counter-rotating modes ( $m < 0$ ) and for co-rotating modes in the *limited range*

$$0 \leq \mu \lesssim \mu_c. \quad (4)$$

Here

$$\mu \equiv \frac{m}{l} \quad (5)$$

is the dimensionless ratio between the azimuthal harmonic index  $m$  and the spheroidal harmonic index  $l$  of the perturbation mode. The critical ratio,  $\mu_c$ , is given by  $\mu_c = \sqrt{\frac{15-\sqrt{193}}{2}} \simeq 0.74$  in the eikonal limit [2, 5]. This critical value of the dimensionless ratio  $\mu$  marks the boundary between perturbations modes (those with  $\mu < \mu_c$ ) which are characterized by *imaginary* values of the angular-eigenvalue  $\delta$  [6, 7] and perturbations modes (those with  $\mu > \mu_c$ ) which are characterized by *real* values of the angular-eigenvalue  $\delta$  [6, 7].

In this Comment we would like to point out that the assertion made in Ref. [2], according to which co-rotating DMs exist *only* in the limited range  $0 \leq \mu \lesssim \mu_c$  [see Eqs. (2) and (4)], is actually erroneous. In particular, we shall show that co-rotating DMs of near-extremal black holes [see Eq. (10) below] actually exist in the *entire* range

$$0 \leq \mu \leq 1 . \quad (6)$$

In fact, Detweiler [8] has obtained an analytic expression for co-rotating DMs of near-extremal black holes which is valid in the regime  $\mu > \mu_c$  [9]:

$$\varpi_n \equiv M(\omega_n - m\Omega_H) = -\frac{e^{\theta/2\delta}}{4m} (\cos \phi + i \sin \phi) \times e^{-\pi n/\delta} , \quad (7)$$

where  $\Omega_H \equiv a/2Mr_+$  is the angular-velocity of the black-hole horizon, and the integer  $n$  is the resonance parameter of the mode. Here we have used the definitions [8]

$$re^{i\theta} \equiv \left[ \frac{\Gamma(2i\delta)}{\Gamma(-2i\delta)} \right]^2 \frac{\Gamma(1/2 + s - im - i\delta)\Gamma(1/2 - s - im - i\delta)}{\Gamma(1/2 + s - im + i\delta)\Gamma(1/2 - s - im + i\delta)} ; \quad \phi \equiv -\frac{1}{2\delta} \ln r . \quad (8)$$

It is worth emphasizing again that the expression (7), originally derived in [8], describes DMs in the  $\mu > \mu_c$  ( $\delta^2 > 0$ ) regime, the regime which was claimed in [2] not to contain DMs.

### III. THE SOURCE OF THE ERRONEOUS CLAIM MADE IN [2]

It is important to understand the reason for the failure of Yang et. al. [2] to observe the DMs (7) of [8] in the regime  $\mu > \mu_c$  [10]. In order to understand the null result of [2] in finding numerically the DMs (7), one should examine the regime of validity of the analyzes presented in [7] and [8].

A careful check of these analyzes reveals that the expression (7) for the black-hole DMs [8] is valid in the regime

$$\tau \ll |\varpi| \ll x \ll 1 , \quad (9)$$

where the dimensionless coordinate  $x \equiv (r - r_+)/r_+$  belongs to an *overlapping* region in which two different expressions for the radial Teukolsky wave function (hypergeometric and confluent hypergeometric functions) can be matched, see [7, 8] for details. Taking cognizance of the inequalities in (9), one realizes that the expression (7) for co-rotating DMs with  $\mu > \mu_c$  is only valid in the regime of near-extremal (rapidly-rotating) black holes.

In particular, since each inequality sign in (9) roughly corresponds to an order-of-magnitude difference between two variables (that is,  $\tau/\varpi \lesssim 10^{-1}$ ,  $\varpi/x \lesssim 10^{-1}$ , and  $x \lesssim 10^{-1}$ ), the expression (7) for the black-hole DMs [8] is not expected to be valid outside the regime [11]

$$\tau \lesssim 10^{-4} . \quad (10)$$

The inequality (10) corresponds to rapidly-rotating black holes with [see Eq. (3)]

$$\frac{a}{M} \gtrsim 1 - 10^{-9} . \quad (11)$$

It is worth noting that the numerical analysis presented in [2] did not explore the deep near-extremal regime (11) of the rotating Kerr black holes [12]. As a consequence, the co-rotating DMs (7) in the regime  $\mu > \mu_c$  have not been observed in the numerical study of [2]. This simple fact has probably led Yang et. al. [2] to the erroneous conclusion that co-rotating DMs are restricted to the limited range  $0 \leq \mu \lesssim \mu_c$ .

#### IV. SUMMARY

It is well-known that rapidly-rotating (near-extremal) black holes are characterized by *two* qualitatively distinct sets of quasinormal resonances: (1) Zero-damping modes (ZDMs), which are characterized by the asymptotic property  $\Im\omega^{\text{ZDM}}(\tau \rightarrow 0) \rightarrow 0$ , and (2) Damped modes (DMs), which are characterized by the asymptotic property  $\Im\omega^{\text{DM}}(\tau \rightarrow 0) \rightarrow \text{finite values}$ .

In this Comment we have refuted the claim made in Ref. [2] that co-rotating DMs of near-extremal black holes are restricted to the limited range  $0 \leq \mu \lesssim \mu_c$  [see Eqs. (2) and (4)]. In particular, we have pointed out that the analytical expression (7), originally derived in [8], describes DMs in the  $\mu > \mu_c$  regime [9], the regime which was claimed in [2] not to contain DMs.

Most importantly, we have emphasized the fact that the analytical expression (7) for the black-hole DMs is not expected to be valid outside the deep near-extremal regime (11) of rapidly-rotating black holes.

Finally, it is worth emphasizing that rapidly-rotating black holes in the regime (11) are probably of no astrophysical relevance [13]. However, these near-extremal black holes are very important from the point of view of quantum field theory. In particular, these black holes play a key role in the conjectured relation between the quantum states of near-extremal black holes and the corresponding quantum states of a two-dimensional conformal field theory [14–17].

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- [1] S. L. Detweiler, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).
  - [2] H. Yang, F. Zhang, A. Zimmerman, D. A. Nichols, E. Berti, and Y. Chen, *Phys. Rev. D* **87**, 041502(R) (2013). See also, H. Yang, A. Zimmerman, A. Zenginoglu, F. Zhang, E. Berti, and Y. Chen, *Phys. Rev. D* **88**, 044047 (2013).
  - [3] For these unique black-hole resonances, see: S. Hod, *Phys. Rev. D* **75**, 064013 (2007) [arXiv:gr-qc/0611004]; S. Hod, *Class. Quant. Grav.* **24**, 4235 (2007) [arXiv:0705.2306]; S. Hod, *Phys. Rev. D* **78**, 084035 (2008) [arXiv:0811.3806]; S. Hod, *Phys. Rev. D* **80**, 064004 (2009) [arXiv:0909.0314].
  - [4] The Bekenstein-Hawking temperatures of Kerr black holes are given by the relation  $T_{\text{BH}} = \tau/8\pi M$ . We shall use units in which  $G = c = \hbar = 1$ .
  - [5] S. Hod, *Phys. Lett. B* **715**, 348 (2012) [arXiv:1207.5282].
  - [6] The parameter  $\delta^2$  is closely related to the angular-eigenvalue of the angular Teukolsky equation, see [7] for details [see, in particular, equations (2.7) and (6.3) of [7]].
  - [7] S. A. Teukolsky and W. H. Press, *Astrophys. J.* **193**, 443 (1974).
  - [8] S. Detweiler, *Astrophys. J.* **239**, 292 (1980).
  - [9] That is, as shown in [8], the expression (7) is valid for co-rotating modes with real  $\delta$  eigenvalues.
  - [10] The failure in [2] to observe these resonances numerically is probably the reason behind the erroneous claim [see Eq. (4)] made in [2].
  - [11] In order to be on the safe side, we have added an extra order of magnitude to the inequality (10).
  - [12] A. Zimmerman (private communication) has kindly updated me that he has not detected in his numerical studies DMs in the regime  $\mu > \mu_c$  for a rapidly-rotating Kerr black hole with  $a/M = 1 - 10^{-9}$ .
  - [13] K. S. Thorne, *Astrophys. J.* **191**, 507 (1974).
  - [14] J. M. Bardeen and G. T. Horowitz, *Phys. Rev. D* **60**, 104030 (1999).
  - [15] M. Guica, T. Hartman, W. Song, and A. Strominger, *Phys. Rev. D* **80**, 124008 (2009).
  - [16] A. Strominger and C. Vafa, *Phys. Lett. B* **379**, 99 (1996).
  - [17] For the physical relevance of these near-extremal black holes to the conjectured universal relaxation bound, see [3] and also: A. Gruzinov, arXiv:gr-qc/0705.1725; A. Pesci, *Class. Quantum Grav.* **24**, 6219 (2007); S. Hod, *Phys. Lett. B* **666** 483 (2008) [arXiv:0810.5419].